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K.V. Karelsky, *Ph.D.,* **A.S. Petrosyan,** *Prof., Sc.D.,* **A.G. Slavin**, *Ph.D.*
Space Research Institute of the Russian Academy of Sciences, Moscow, Russia

FINITE–DIFFERENCE PRESENTATION OF THE CORIOLIS FORCE FOR FLOWS OF ROTATING SHALLOW WATER

Abstract. *In the given work the finite-difference presentation is proposed, that describes the Coriolis force in numerical Godunov-type methods for rotating shallow water flows. The finite-difference schemes are offered for simulating the flows, both on a smooth underlying surface, and for underlying surface of arbitrary profile. The Coriolis force effect is simulated by introducing the fictitious non-stationary boundary. For numerical approximation of source terms, caused by inhomogeneity of underlying surface and Coriolis force effect, the quasi-two-layer model of fluid flow over a stepwise boundary is applied, that takes into account hydrodynamic features. The calculations showing the efficiency of the proposed method are carried out.*

Keywords: *shallow water, rotation, Coriolis force, quasi-two-layer method, arbitrary underlying surface.*

1 Introduction

The Euler equations, which most completely describe the hydrodynamics of natural and laboratory flows of ideal fluid, are so complicated, that in the presence of a complex boundary, even under assumption of non-compressibility, barotropicity and the absence of rotation, they can not be numerically integrated in the problems with rather strong change of underlying surface geometry. The necessity of reducing the initial equations in a class of problems with free surface has compelled Stoker to construct the lower-order mathematical model that was later called the «shallow water» model. This action was stipulated by the fact, that the reduction was performed in the model by asymptotic expansion over a small parameter determined by the ratio of fluid depth to the characteristic linear value, under assumption of hydrostatic character of pressure distribution and weakness of horizontal velocities variation along the lines collinear to the vector of gravity. The obtained equations, in virtue of their nonlinearity and underlying surface complexity, also occurred to be rather complicated for constructing general analytical solutions. However, they can be successfully integrated numerically [1,10].

The main difficulty of numerical simulation of an inhomogeneous system of shallow water equations (SWE) consists in its non-divergence character determined by inhomogeneity of the right-hand side of the momentum conservation equations. The presence of a non-divergent term induces, from the physical point of view, highly nonlinear effects caused by stepwise change of hydrodynamic quantities in the areas of its sharp change. In studying small-scale natural flows under the shallow water approximation, where the Coriolis force effect is insignificant, the numerical methods have been developed and are being effectively used now. These methods were initially developed for solving the gas dynamics equations, since the shallow water equations for a horizontal bed are similar to the barotropic gas flows with the adiabatic index equal to two. The problem of numerical integration of SWE in the presence of underlying surface inhomogeneity (Saint-Venant’s equations) is reduced, with some reservations, to a similar problem of gas flow in a converging nozzle [2].

The presence of well-developed and approved numerical techniques, along with repeatedly tested software implementation, made especially attractive the reduction of the solution of the problem on rotating "shallow water” over a smooth underlying surface to the solution of the problem on shallow water flows over a complex non-stationary boundary.

The possibility of simulating the shallow water by introducing the fictitious non-stationary boundary has been used for a long time for constructing numerical models in the geophysical hydrodynamics [1,8,9]. The fruitfulness of borrowing the methods, developed for complex underlying surfaces, in solving the problems with the Coriolis force, perhaps, cause no doubts in anybody. However, the influence of orography, obviously, directly results in the existence of underlying surface's work over the fluid flow, whereas the Coriolis force, by itself, can not commit any work. In this work we discuss the physical interpretation of aforementioned formal borrowing and analyze the boundaries of applicability of the given approach, in general, for splitting difference schemes. The main problem in constructing a splitting difference scheme consists in the necessity of statement and solution of a one-dimensional problem, which does not have any physical equivalent for finite time intervals. Unfortunately, the most obvious way of neglecting one of spatial coordinates does not solve the problem completely. Really, refusal from one of spatial variables in solution of an essentially two-dimensional problem, results in violating the momentum conservation law, which, in its turn, causes the necessity of introducing some fictitious work for compensating mentioned violations, in spite of all non-physical character of such a compensation.

In this work, for calculating the shallow water flows over an arbitrary surface in the Coriolis force presence, the modernized Godunov method is proposed, which is adapted to flow parameters. The proposed method belongs to the family of methods based on solution of the problem of breaking an arbitrary discontinuity. This method is based on successive solution of classical shallow water equations on a smooth plane by using the Godunov method with allowance for the vertical inhomogeneity of flow in calculating the flows through the boundaries of cells adjacent to stepwise boundaries. The accounting for the vertical inhomogeneity is provided by using the Riemann problem solution on a step based on the quasi-two-layer shallow water model developed in [4,5]. The distinctive feature of this model is separation of a studied flow into two layers in calculating flow quantities near each step, with improving the approximation of initial three-dimensional Euler's equations. The unambiguity of such a separation into two layers is provided by the uniqueness of solution of the Dirichlet problem for finding this boundary. Adapting to the flow parameters, this method allows one to take into account the features of fluid flow at each point of space and at any time instant.

The method proposed in the given work makes it possible to visually represent the features of a splitting approach to numerical simulation as a whole and, thus, provides physical considerations for updating the stability criteria in the finite-difference implementation. Within the framework of the proposed method, the structure of the solution inside the considered spatial-temporal region for the depth and one of components of the velocity vector becomes known, that allows one, under assumption of stability, to re-calculate the transversal velocity and, thus, to minimize the parasitic phenomena, which are caused mathematically by refusal from integration of the equation for the transversal component of a velocity vector, that induces, from the physical point of view, the non-compensated work of the Coriolis force inside a flow.

2 The Godunov method for the equations of rotating shallow water over an underlying surface of arbitrary profile within the framework of the quasi-two-layer approximation

Replacement of the terms, which are responsible for the Coriolis force effect:

$$g\frac{dk}{dx} = -fv, \quad g\frac{dk}{dy} = fu\,, \tag{1}$$

where g is the free falling acceleration, $u(x,y,t)$ is the averaged-over-depth horizontal component of velocity in the x direction, $v(x,y,t)$ is the averaged-over-depth horizontal component of velocity in the y direction, $f = 2\Omega\sin\phi$ is the Coriolis parameter, where $\Omega=7.292\times10^{-5}$ s^{-1} is the frequency of rotation of the Earth, ϕ is the fixed latitude.

The system of equations of rotating shallow water over an underlying surface of arbitrary profile $z = b(x,y)$, with regard replacement (1) are written as:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial(hu)}{\partial x} + \dfrac{\partial(hv)}{\partial y} = 0 \\ \dfrac{\partial(hu)}{\partial t} + \dfrac{\partial\left(hu^2 + gh^2/2\right)}{\partial x} + \dfrac{\partial huv}{\partial y} = -gh\dfrac{db}{dx} - gh\dfrac{dk}{dx}, \\ \dfrac{\partial(hv)}{\partial t} + \dfrac{\partial\left(hv^2 + gh^2/2\right)}{\partial y} + \dfrac{\partial hvu}{\partial x} = -gh\dfrac{db}{dy} - gh\dfrac{dk}{dy} \end{cases} \quad (2)$$

where $h(x,y,t)$ is the depth of fluid or gas. The influence of terms $-gh\,dk/dx$ and $-gh\,dk/dy$, as compared to the terms responsible for underlying surface, $-gh\,db/dx$ and $-gh\,db/dy$, at each time step is an order lower; therefore, for obtaining the adequate flow picture it is necessary to keep the calculation accuracy as corresponding to the minimum effects determined by the Coriolis force.

For constructing a grid, the effective boundary is approximated by the piece-constant function. Applying the integral conservation laws to each cell and supposing that on cell's boundary the values of all hydrodynamic parameters remain unchanged during the time step of integration, we obtain the difference scheme:

$$H_{x,y}^{t+1} = H_{x,y}^{t} + \tau\times\left(\frac{H_{x-1/2,y}^{t}U_{x-1/2,y}^{t} - H_{x+1/2,y}^{t}U_{x+1/2,y}^{t}}{X} + \frac{H_{x,y-1/2}^{t}V_{x,y-1/2}^{t} - H_{x,y+1/2}^{t}V_{x,y+1/2}^{t}}{Y} \right), \quad (3)$$

$$U_{x,y}^{t+1} = \tau\times\left(\frac{g\left(H_{x-1/2,y}^{t} + i_{sur}\times\left(B_{x,y} - B_{x-1,y}\right) + i_{cor}\times\left(K_{x,y} - K_{x-1,y}\right)\right)^2}{2} + H_{x-1/2,y}^{t}\left(U_{x-1/2,y}^{t}\right)^2 - \right.$$

$$\left. - \frac{g\left(H_{x+1/2,y}^{t} + i_{sur}\times\left(B_{x+1,y} - B_{x,y}\right) + i_{cor}\times\left(K_{x+1,y} - K_{x,y}\right)\right)^2}{2} - H_{x+1/2,y}^{t}\left(U_{x+1/2,y}^{t}\right)^2 \right) \Big/ XH_{x,y}^{t+1} +$$

$$+\tau\times\left(H_{x,y-1/2}^{t}U_{x,y-1/2}^{t}V_{x,y-1/2}^{t} - H_{x,y+1/2}^{t}U_{x,y+1/2}^{t}V_{x,y+1/2}^{t}\right) \Big/ YH_{x,y}^{t+1} + \frac{H_{x,y}^{t}U_{x,y}^{t}}{H_{x,y}^{t+1}},$$

$$V_{x,y}^{t+1}=\tau\times\left(\frac{g\left(H_{x,y-1/2}^{t}+i_{sur}\times\left(B_{x,y}-B_{x,y-1}\right)+i_{cor}\times\left(K_{x,y}-K_{x,y-1}\right)\right)^{2}}{2}+H_{x,y-1/2}^{t}\left(U_{x,y-1/2}^{t}\right)^{2}-\right.$$

$$\left.-\frac{g\left(H_{x,y+1/2}^{t}+i_{sur}\times\left(B_{x,y+1}-B_{x,y}\right)+i_{cor}\times\left(K_{x,y+1}-K_{x,y}\right)\right)^{2}}{2}-H_{x,y+1/2}^{t}\left(U_{x,y+1/2}^{t}\right)^{2}\right)\Big/YH_{x,y}^{t+1}+$$

$$+\tau\times\left(H_{x-1/2,y}^{t}U_{x-1/2,y}^{t}V_{x-1/2,y}^{t}-H_{x+1/2,y}^{t}U_{x+1/2,y}^{t}V_{x+1/2,y}^{t}\right)\Big/XH_{x,y}^{t+1}+\frac{H_{x,y}^{t}V_{x,y}^{t}}{H_{x,y}^{t+1}}$$

where τ is a step in time, *X* and *Y* are steps in space $H_{x,y}$ – the depth of fluid, $\left(V_{x,y},U_{x,y}\right)$ – the velocity vector of fluid, $B_{x,y}$ – the height of underlying surface, and $K_{x,y}$ – the height of a fictitious boundary. Subscripts *x, y* designate the values of a function related to the center of masses of a cell with number *(x, y)*. Semi-subscripts $x\pm1/2, y\pm1/2$ designate the values of quantities at the boundary between cells with numbers *x*, $x\pm1$, and *y*, $y\pm1$, respectively. Superscript *t* designates the number of a step in time, *f* is the Coriolis parameter, i_{sur} and i_{cor} assume the values: either 0 – in the case of a negative drop of heights of an underlying surface and fictitious boundary, respectively, or 1 – in the case of positive drop. If the drop of heights is absent, the depths $H_{x\pm1/2,y}^{t}, H_{x,y\pm1/2}^{t}$ and velocities $U_{x\pm1/2,y}^{t}, \mathrm{V}_{x,y\pm1/2}^{t}$ of fluids are calculated by the solution of the usual Riemann problem on a smooth plane, and difference scheme (3), identically transfer into the standard Godunov's difference scheme for the classical equations of shallow water over a smooth bed. In other case, the depths $H_{x\pm1/2,y}^{t}, H_{x,y\pm1/2}^{t}$ and velocities $U_{x\pm1/2,y}^{t}, \mathrm{V}_{x,y\pm1/2}^{t}$ of fluids are calculated with the help of quasi-two-layer model [4,5].

3 Numerical simulation results

The quasi-two-layer method of the second order accuracy is applied for numerical modelling on space and time. Increase of accuracy order on space is reached by applying the piecewise-linear reconstruction to the distribution of functions value in a cell with the use of the minmod limiter (4) suggested for the first time by Kolgan [6] for the solutions of accuracy problems of Godunov type: methods in gas-dynamics:

$$W_{x}^{t}=\min\operatorname{mod}\left(\frac{F_{x+1}^{t}-F_{x}^{t}}{\Delta x},\frac{F_{x}^{t}-F_{x-1}^{t}}{\Delta x}\right)$$
$$\min\operatorname{mod}(a,b)=\alpha\frac{1}{2}\left(signa+signb\right)\min\left(|a|,|b|\right)\quad ,\text{where } F_{x}^{t}\equiv\begin{pmatrix}H_{x}^{t}\\ U_{x}^{t}\\ V_{x}^{t}\end{pmatrix},\ \alpha=0.72. \tag{4}$$

The second order of accuracy on time is reached by application of two-step-by-step algorithm predictor – corrector. At a stage predictor there were found auxiliary values of required sizes for the whole step on time with the help of quasi-two-layer algorithm of the first order of the accuracy. These auxiliary values used for finding of values on an intermediate step on time by using arithmetic averaging with values of the previous time step.

On a step the corrector the given sizes is reconstructed on space: $F_x^{t+\frac{\tau}{2}} + \frac{1}{2}\Delta x W_x^t$, $F_{x+1}^{t+\frac{\tau}{2}} - \frac{1}{2}\Delta x W_{x+1}^t$, accordingly at the left and at the right of the grid $x+1/2$. Next the values fluid variables on border of cells corresponding to an intermediate time layer are found. The rectangular grid of the size 200×10 cells is used.

The classical Rossby problem was simulated as the test one. The initial disturbance was considered: $\begin{cases} h(x,0) = h_0 \\ u(x,0) = 0 \\ v(x,0) = Vv_{jet}(x) \end{cases}$, where h_0 is the initial depth of rest, V is the characteristic scale of velocity, $v_{jet}(x)$ is the normalized profile, that is specified as follows: $v_{jet}(x) = \frac{(1+\tanh(4x/L+2))(1-\tanh(4x/L-2))}{(1+\tanh(2))^2}$.

The form of a profile is presented in Fig. 1, where L is the characteristic scale of disturbance. Characteristic parameters g, h_0, f were fixed. The characteristic scale of velocity V and the characteristic scale of disturbance L were calculated from two dimensionless parameters: the Rossby-Kibel (Ro) and Burgers (Bu) numbers: $\mathrm{Ro} = \frac{V}{fL}$, $\mathrm{Bu} = \frac{R_d^2}{L^2}$, where R_d is the deformation radius: $R_d = \frac{\sqrt{gh_0}}{f}$. The characteristic time scale is specified by the following formula: $T_f = \frac{2\pi}{f}$. The results of evolution of depth h_0, in the case of $\mathrm{Ro} = 1$, $\mathrm{Bu} = 0.25$, are presented below.

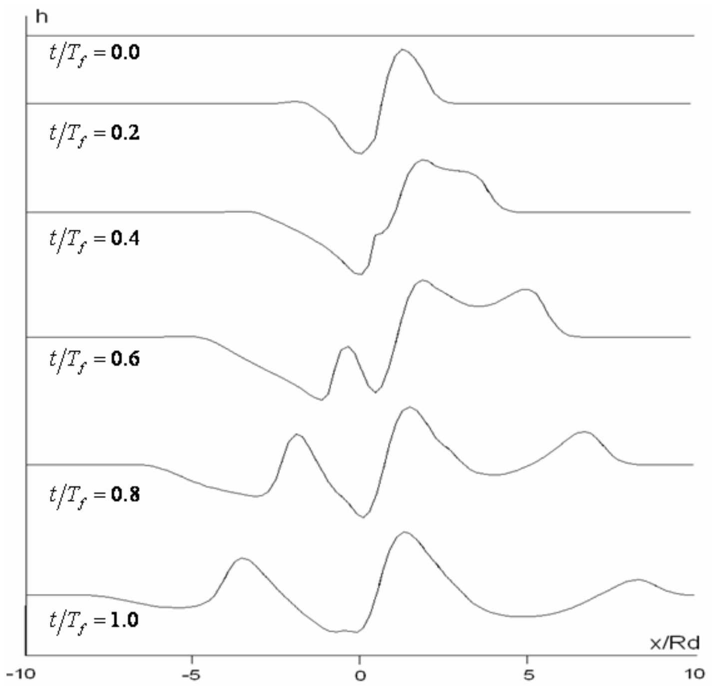

Fig. 1 – Evolution of acoustic–gravitational waves propagation, as a result of effect of the initial disturbance $Vv_{jet}(x)$ with using the quasi-two-layer method.

Figure 1 shows the evolution obtained bymeans of the quasi-two-layer model.

One can see good coincidence of characteristic peaks of running-away acoustic-gravitational waves and the central balanced part with results presented in paper [1]. This testifies to the efficiency of using the quasi-two-layer model in the description of large-scale geophysical phenomena.

Figure 2 shows the comparison of potential vorticity values at the initial ($t = 0T_f$) and final ($t = 16T_f$) time instants for the classical Rossby problem and $\mathrm{Ro} = 1$, $\mathrm{Bu} = 0.5$.

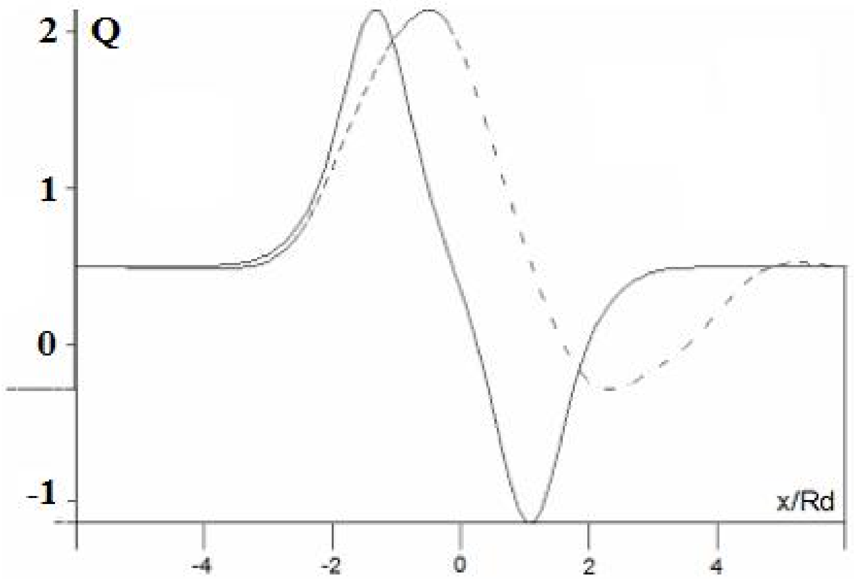

Fig. 2 – Potential vorticity at the initial (solid line, $t = 0T_f$) and final (dashed line, $t = 16T_f$) time instants.

One can see that the invariant Q – the potential vortex – conserves with time. Note that real time of the process equals 12 days, approximately. It is seen from presented plot that the maximum of a function is shifted to the anticyclonic region, and the minimum of the potential vorticity increases with time. The given results are determined by purely nonlinear effects and well agree with those obtained in paper [7].

Also for demonstrating the workability of the numerical method, the simulation of the shallow water flow, in the presence of Coriolis force, over the mountain-like underlying surface was carried out. Typical parameters of the problem were: the linear dimensions 10^6 by 10^6 m, the mountain height was $1,2 \cdot 10^3$ m, the fluid depth was $2 \cdot 10^3$ m. The initial wind parameters were: $u = 0$ m/s, $v = 20$ m/s. Numerical method used for calculations is Godunov-type method. The idea of Godunov-type methods consists in splitting the solution of a multi-dimensional problem into a set of sub-problems, which arise after dividing into cells the calculated domain, and writing down the relevant integral relations for all elements (cells), by means of which cubing was performed. Numerical grid used in test: 100 by 100 cells. As a result, the characteristic time of one revolution of a system as a whole was found to be 25 hours, which corresponds, to a sufficient accuracy, to the natural phenomenon (the characteristic time of one revolution of a system as a whole for the geophysical dynamics problems equals one day [3]). Figure 3 shows the flow evolution during 25 hours.

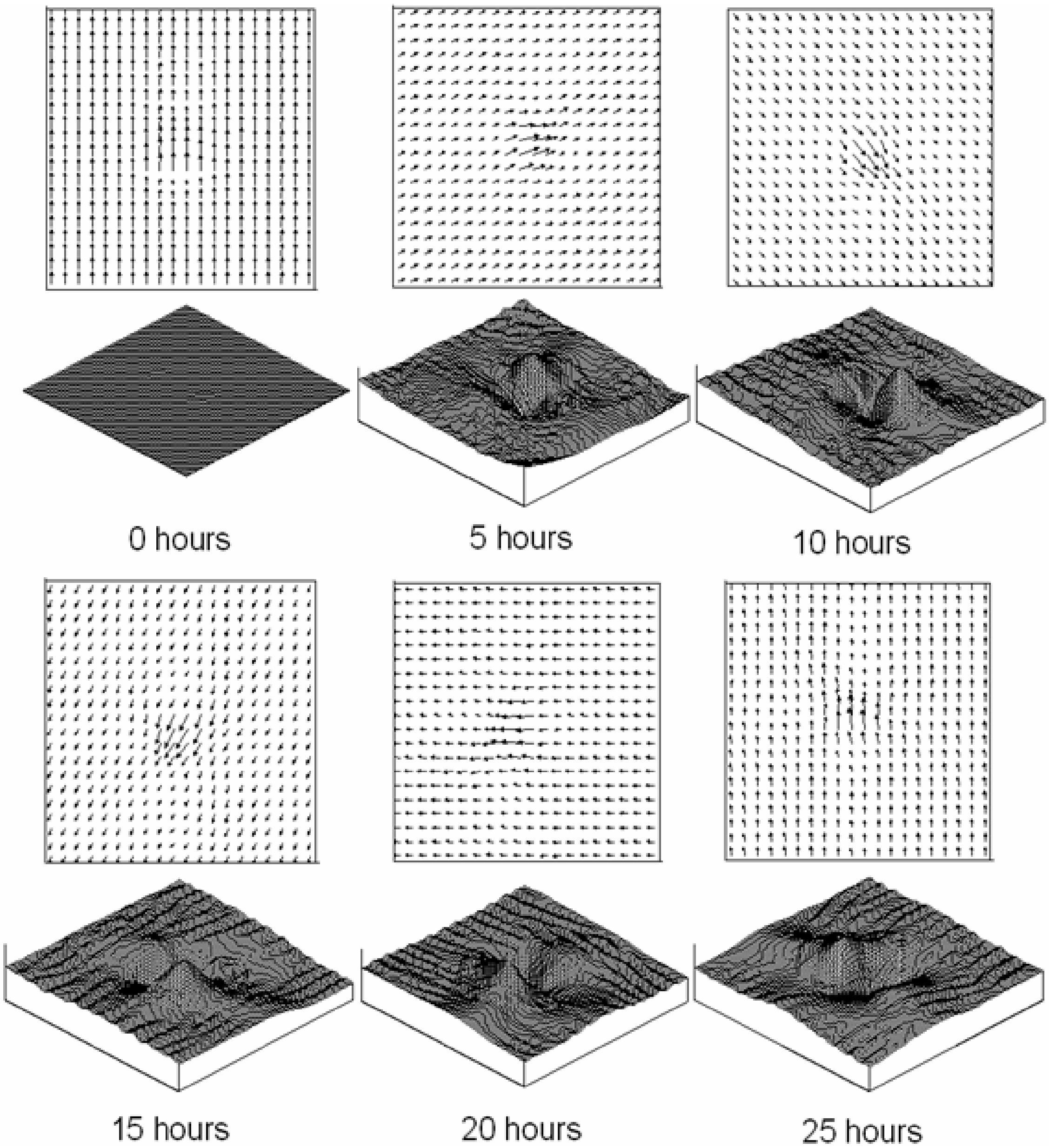

Fig.3 Evolution of fluid/gas flow under the Coriolis force effect over a mountain; the upper plots – fields of velocities, the lower ones – the free surface.

4 Conclusion

The given work proposes the finite-difference model that allows one to describe the Coriolis force in the numerical Godunov-type methods for rotating shallow water flows. On the basis of the proposed presentation, the finite-volume algorithm is developed, both for a smooth boundary, and for a boundary of arbitrary type. The method is based on the presentation of arbitrary underlying surface and Coriolis force by a complex non-stationary stepwise boundary. The boundaries of applicability of the given presentation are discussed from the viewpoint of finite-volume numerical methods. The finite-difference approximation of the effective complex non-stationary surface is performed on the basis of a quasi-two-layer shallow water model, which is more correct, in relation to the initial system of Euler's equations, as compared to the classical single-layer models. The proposed presentation adequately describes the features of nonlinear processes caused by the Coriolis force in numerical Godunov-type methods, because it correctly reflects the nonlinear structure of flows near the features arising after digitization of an effective boundary. The basic advantage of the proposed method is revealed, that allows one to more correctly describe the transversal component of the velocity vector and, thus, to minimize the calculation error induced by an essentially two-dimensional character of statements of problems for rotating fluid. The implementation of the mentioned advantage will be considered in a separate paper.

The workability of the method was confirmed by the numerical experiment on simulating the classical geostrophic adaptation problem, known as the Rossby problem, and by calculation of rotating shallow water over an underlying surface of parabolic profile.

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Конечно-разностное представление силы Кориолиса для течений вращающейся мелкой воды

Аннотация. *В данной работе предложено конечно-разностное представление, описывающее силу Кориолиса в численных методах Годуновского типа для течений вращающейся мелкой воды. Предложены конечно-разностные схемы для моделирования течений, как на ровной подстилающей поверхности, так и для подстилающей поверхности произвольного профиля. Влияние силы Кориолиса моделируется введением фиктивной нестационарной границы. Для численной аппроксимации источниковых слагаемых, вследствие неоднородности подстилающей поверхности и влияния силы Кориолиса, применена квазидвухслойная модель течения жидкости над ступенчатой границей, учитывающая гидродинамические особенности. Выполнены расчеты, показывающие эффективность предложенного метода.*

Ключевые слова: *уравнения мелкой воды, вращение, сила Кориолиса, квази-двуслойный метод, произвольная подстилающая поверхность*